

Some Properties of the Nil-Graphs of Ideals of Commutative Rings ^{*†}

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Abstract

Let R be a commutative ring with identity and $\text{Nil}(R)$ be the set of nilpotent elements of R . The nil-graph of ideals of R is defined as the graph $\mathbb{AG}_N(R)$ whose vertex set is $\{I : (0) \neq I \triangleleft R \text{ and there exists a non-trivial ideal } J \text{ such that } IJ \subseteq \text{Nil}(R)\}$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq \text{Nil}(R)$. Here, we study conditions under which $\mathbb{AG}_N(R)$ is complete or bipartite. Also, the independence number of $\mathbb{AG}_N(R)$ is determined, where R is a reduced ring. Finally, we classify Artinian rings whose nil-graphs of ideals have genus at most one.

1. Introduction

When one assigns a graph with an algebraic structure, numerous interesting algebraic problems arise from the translation of some graph-theoretic parameters such as clique number, chromatic number, diameter, radius and so on. There are many papers in this topic, see for example [5], [8] and [12]. Throughout this paper, all rings are assumed to be non-domain commutative rings with identity. By $\mathbb{I}(R)$ ($\mathbb{I}(R)^*$) and $\text{Nil}(R)$, we denote the set of all proper (non-trivial) ideals of R and the nil-radical of

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R , respectively. The set of all maximal and minimal prime ideals of R are denoted by $\text{Max}(R)$ and $\text{Min}(R)$, respectively. The ring R is said to be *reduced*, if it has no non-zero nilpotent element.

Let G be a graph. The degree of a vertex x of G is denoted by $d(x)$. The graph G is said to be *r-regular*, if the degree of each vertex is r . The *complete graph* with n vertices, denoted by K_n , is a graph in which any two distinct vertices are adjacent. A *bipartite graph* is a graph whose vertices can be divided into two disjoint parts U and V such that every edge joins a vertex in U to one in V . It is well-known that a bipartite graph is a graph that does not contain any odd cycle. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. If the size of one of the parts is 1, then it is said to be a *star graph*. A *tree* is a connected graph without cycles. Let S_k denote the sphere with k handles, where k is a non-negative integer, that is, S_k is an oriented surface of genus k . The *genus* of a graph G , denoted by $\gamma(G)$, is the minimal integer n such that the graph can be embedded in S_n . A genus 0 graph is called a *planar graph*. It is well-known that

$$\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil \text{ for all } n \geq 3,$$

$$\gamma(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil, \text{ for all } n \geq 2 \text{ and } m \geq 2.$$

For a graph G , the *independence number* of G is denoted by $\alpha(G)$. For more details about the used terminology of graphs, see [13].

We denote the annihilator of an ideal I by $\text{Ann}(I)$. Also, the ideal I of R is called an *annihilating-ideal* if $\text{Ann}(I) \neq (0)$. The notation $\mathbb{A}(R)$ is used for the set of all annihilating-ideals of R . By the *annihilating-ideal graph* of R , $\mathbb{AG}(R)$, we mean the graph with vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. Some properties of this graph have been studied in [1, 2, 3, 5, 6]. In [12], the authors have introduced another kind of graph, called the nil-graph of ideals. The nil-graph of ideals of R is defined as the graph $\mathbb{AG}_N(R)$ whose vertex set is $\{I : (0) \neq I \triangleleft R \text{ and there exists a non-trivial ideal } J \text{ such that } IJ \subseteq \text{Nil}(R)\}$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq \text{Nil}(R)$. Obviously, our definition is slightly different from the one defined by Behboodi and Rakeei in [5] and it is easy to see that the usual annihilating-ideal graph $\mathbb{AG}(R)$ is a subgraph of $\mathbb{AG}_N(R)$. In [12], some basic properties of nil-graph of ideals have been studied. In this article, we continue the study of the nil-graph of ideals. In Section 2, the necessary and sufficient conditions, under which the nil-graph of a ring is complete or bipartite, are found. Section 3 is devoted to the studying of independent sets in nil-graph ideals. In

Section 4, we classify all Artinian rings whose nil-graphs of ideals have genus at most one.

2. When Is the Nil-Graph of Ideals Complete or Bipartite?

In this section, we study conditions under which the nil-graph of ideals of a commutative ring is complete or complete bipartite. For instance, we show that if R is a Noetherian ring, then $\mathbb{A}\mathbb{G}_N(R)$ is a complete graph if and only if either R is Artinian local or $R \cong F_1 \times F_2$, where F_1 and F_2 are fields. Also, it is proved that if $\mathbb{A}\mathbb{G}_N(R)$ is bipartite, then $\mathbb{A}\mathbb{G}_N(R)$ is complete bipartite. Moreover, if R is non-reduced, then $\mathbb{A}\mathbb{G}_N(R)$ is star and $\text{Nil}(R)$ is the unique minimal prime ideal of R .

We start with the following theorem which can be viewed as a consequence of [12, Theorem 5] (Here we prove it independently). Note that it is clear that if R is a reduced ring, then $\mathbb{A}\mathbb{G}_N(R) \cong \mathbb{A}\mathbb{G}(R)$.

Theorem 1. *Let R be a Noetherian ring. Then $\mathbb{A}\mathbb{G}_N(R)$ is a complete graph if and only if either R is Artinian local or $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof. First suppose that $\mathbb{A}\mathbb{G}_N(R)$ is complete. If R is reduced, then by [5, Theorem 2.7], $R \cong F_1 \times F_2$. Thus we can suppose that $\text{Nil}(R) \neq (0)$. We continue the proof in the following two cases:

Case 1. R is a local ring with the unique maximal ideal \mathfrak{m} . Since R is non-reduced, by Nakayama's Lemma (see [4, Proposition 2.6]), \mathfrak{m} and \mathfrak{m}^2 are two distinct vertices of $\mathbb{A}\mathbb{G}_N(R)$. Thus $\mathfrak{m}^3 \subseteq \text{Nil}(R)$ and so R is an Artinian local ring.

Case 2. R has at least two maximal ideals. First we show that R has exactly two maximal ideals. Suppose to the contrary, $\mathfrak{m}, \mathfrak{n}$ and \mathfrak{p} are three distinct maximal ideals of R . Since $\mathbb{A}\mathbb{G}_N(R)$ is complete, we deduce that $\mathfrak{m}\mathfrak{n} \subseteq \text{Nil}(R) \subseteq \mathfrak{p}$, a contradiction. Thus R has exactly two maximal ideals, say \mathfrak{m} and \mathfrak{p} . Now, we claim that both \mathfrak{m} and \mathfrak{p} are minimal prime ideals. Since \mathfrak{m} and \mathfrak{p} are adjacent, we conclude one of the maximal ideals, say \mathfrak{p} , is a minimal prime ideal of R . Now, suppose to the contrary, \mathfrak{m} properly contains a minimal prime ideal \mathfrak{q} of R . Since $\mathfrak{m}\mathfrak{p} \subseteq \mathfrak{q}$, we get a contradiction. So the claim is proved. Thus R is Artinian. Hence by [4, Theorem 8.7], we have $R \cong R_1 \times R_2$, where R_1 and R_2 are Artinian local rings. By contrary and with no loss of generality, suppose that R_1 contains a non-trivial ideal, say I . Then the vertices $I \times R_2$ and $(0) \times R_2$ are not adjacent, a contradiction. Thus $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.

Conversely, if $R \cong F_1 \times F_2$, where F_1 and F_2 are fields, then it is clear that $\mathbb{A}\mathbb{G}_N(R) \cong K_2$. Now, suppose that (R, \mathfrak{m}) is an Artinian local ring. Since \mathfrak{m} is nilpotent, it follows that $\mathbb{A}\mathbb{G}_N(R)$ is complete. \square

The following example shows that Theorem 1 does not hold for non-Noetherian rings.

Example 2. Let $R = \frac{k[x_i: i \geq 1]}{(x_i^2: i \geq 1)}$, where k is a field. Then R is not Artinian and $\mathbb{A}\mathbb{G}_N(R)$ is a complete graph.

Remark 3. Let R be a ring. Every non-trivial ideal contained in $\text{Nil}(R)$ is adjacent to every other vertex of $\mathbb{A}\mathbb{G}_N(R)$. In particular, if R is an Artinian local ring, then $\mathbb{A}\mathbb{G}_N(R)$ is a complete graph.

The next result shows that nil-graphs, whose every vertices have finite degrees, are finite graphs.

Theorem 4. *If every vertex of $\mathbb{A}\mathbb{G}_N(R)$ has a finite degree, then R has finitely many ideals.*

Proof. First suppose that R is non-reduced. Since $d(\text{Nil}(R)) < \infty$, the assertion follows from Remark 3. Thus we can assume that R is reduced. Choose $0 \neq x \in Z(R)$. Since $d(Rx) < \infty$ and Rx is adjacent to every ideal contained in $\text{Ann}(x)$, we deduce that $\text{Ann}(x)$ is an Artinian R -module. Similarly, one can show that Rx is an Artinian R -module. Now, the R -isomorphism $Rx \cong \frac{R}{\text{Ann}(x)}$ implies that R is an Artinian ring. Now, since R is reduced, [4, Theorem 8.7] implies that R is a direct product of finitely many fields and hence we are done. \square

The next result gives another condition under which $\mathbb{A}\mathbb{G}_N(R)$ is complete.

Theorem 5. *If $\mathbb{A}\mathbb{G}_N(R)$ is an r -regular graph, then $\mathbb{A}\mathbb{G}_N(R)$ is a complete graph.*

Proof. If $\text{Nil}(R) \neq (0)$, then by Remark 3, there is nothing to prove. So, suppose that R is reduced. Since $\mathbb{A}\mathbb{G}_N(R)$ is an r -regular graph, Theorem 4 and [4, Theorem 8.7] imply that $R \cong F_1 \times \cdots \times F_n$, where $n \geq 2$ and each F_i is a field. It is not hard to check that every ideal $I = I_1 \times \cdots \times I_n$ of R has degree $2^{n_I} - 1$, where $n_I = |\{i : 1 \leq i \leq n \text{ and } I_i = (0)\}|$. Let $I = F_1 \times (0) \times \cdots \times (0)$ and $J = F_1 \times \cdots \times F_{n-1} \times (0)$. Then we have $d(I) = 2^{n-1} - 1$ and $d(J) = 1$. The r -regularity of $\mathbb{A}\mathbb{G}_N(R)$ implies that $2^{n-1} - 1 = 1$ and so $n = 2$. Therefore $R \cong F_1 \times F_2$, as desired. \square

In the rest of this section, we study bipartite nil-graphs of ideals of rings.

Theorem 6. *Let R be a ring such that $\mathbb{A}\mathbb{G}_N(R)$ is bipartite. Then $\mathbb{A}\mathbb{G}_N(R)$ is complete bipartite. Moreover, if R is non-reduced, then $\mathbb{A}\mathbb{G}_N(R)$ is star and $\text{Nil}(R)$ is the unique minimal prime ideal of R .*

Proof. If R is reduced, then by [6, Corollary 2.5], $\mathbb{A}\mathbb{G}_N(R)$ is a complete bipartite graph. Now, suppose that R is non-reduced. Then by Remark 3, $\mathbb{A}\mathbb{G}_N(R)$ is a star graph. So, by Remark 3, either $\text{Nil}(R)$ is a minimal ideal or R has exactly two ideals. In the latter case, R is an Artinian local ring and so $\text{Nil}(R)$ is the unique minimal prime ideal of R . Thus we can assume that $\text{Nil}(R) = (x)$ is a minimal ideal of R , for some $x \in R$. To complete the proof, we show that R has exactly one minimal prime ideal. Suppose to the contrary, \mathfrak{p}_1 and \mathfrak{p}_2 are two distinct minimal prime ideals of R . Choose $z \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ and set $S_1 = R \setminus \mathfrak{p}_1$ and $S_2 = \{1, z, z^2, \dots\}$. If $0 \notin S_1 S_2$, then by [11, Theorem 3.44], there exists a prime ideal \mathfrak{p} in R such that $\mathfrak{p} \cap S_1 S_2 = \emptyset$ and hence $\mathfrak{p} = \mathfrak{p}_1$, a contradiction. So, $0 \in S_1 S_2$. Therefore, there exist positive integer k and $y \in R \setminus \mathfrak{p}_1$ such that $yz^k = 0$. Consider the ideals $(x), (y)$ and (z^k) . This is clear that $(x), (y)$ and (z^k) are three distinct vertices which form a triangle in $\mathbb{A}\mathbb{G}_N(R)$, a contradiction. \square

The following corollary is an immediate consequence of Theorem 6 and Remark 3.

Corollary 7. *If $\mathbb{A}\mathbb{G}_N(R)$ is a tree, then $\mathbb{A}\mathbb{G}_N(R)$ is a star graph.*

We finish this section with the next corollary.

Corollary 8. *Let R be an Artinian ring. Then $\mathbb{A}\mathbb{G}_N(R)$ is bipartite if and only if $\mathbb{A}\mathbb{G}_N(R) \cong K_n$, where $n \in \{1, 2\}$.*

Proof. Let R be an Artinian ring and $\mathbb{A}\mathbb{G}_N(R)$ be bipartite. Then by Theorem 6, $\mathbb{A}\mathbb{G}_N(R)$ is complete bipartite. If R is local, then Remark 3 implies that $\mathbb{A}\mathbb{G}_N(R)$ is complete. Since $\mathbb{A}\mathbb{G}_N(R)$ is complete bipartite, we deduce that $\mathbb{A}\mathbb{G}_N(R) \cong K_n$, where $n \in \{1, 2\}$. Now, suppose that R is not local. Then by [4, Theorem 8.7], there exists a positive integer n such that $R \cong R_1 \times \dots \times R_n$, where every R_i is an Artinian local ring. Since $\mathbb{A}\mathbb{G}_N(R)$ contains no odd cycle, it follows that $n = 2$. To complete the proof, we show that both R_1 and R_2 are fields. By contrary and with no loss of generality, suppose that R_1 contains a non-trivial ideal, say I . Then it is not hard to check that $R_1 \times (0), I \times (0)$ and $(0) \times R_2$ forms a triangle in $\mathbb{A}\mathbb{G}_N(R)$, a contradiction. The converse is trivial. \square

3. The Independence Number of Nil-Graphs of Ideals

In this section, we use the maximal intersecting families to obtain a low bound for the independence number of nil-graphs of ideals. Let $R \cong R_1 \times R_2 \times \cdots \times R_n$,

$$\mathcal{T}(R) = \{(0) \neq I = I_1 \times I_2 \times \cdots \times I_n \triangleleft R \mid \forall 1 \leq k \leq n : I_k \in \{(0), R_k\}\};$$

and denote the induced subgraph of $\mathbb{A}\mathbb{G}_N(R)$ on $\mathcal{T}(R)$ by $G_{\mathcal{T}}(R)$.

Proposition 9. *If $R \cong R_1 \times R_2 \times \cdots \times R_n$ is a ring, then $\alpha(G_{\mathcal{T}}(R)) = 2^{n-1}$.*

Proof. For every ideal $I = I_1 \times I_2 \times \cdots \times I_n$, let

$$\Delta_I = \{k \mid 1 \leq k \leq n \text{ and } I_k = R_k\};$$

Then two distinct vertices I and J in $G_{\mathcal{T}}(R)$ are not adjacent if and only if $\Delta_I \cap \Delta_J \neq \emptyset$. So, there is a one to one correspondence between the independent sets of $G_{\mathcal{T}}(R)$ and the set of families of pairwise intersecting subsets of the set $[n] = \{1, 2, \dots, n\}$. So, [10, Lemma 2.1] completes the proof. \square

Using [4, Theorem 8.7], we have the following immediate corollary.

Corollary 10. *Let R be an Artinian with n maximal ideals. Then $\alpha(\mathbb{A}\mathbb{G}_N(R)) \geq 2^{n-1}$; moreover, the equality holds if and only if R is reduced.*

Lemma 11. [9, Proposition 1.5] *Let R be a ring and $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be a finite set of distinct minimal prime ideals of R . Let $S = R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$. Then $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$.*

Proposition 12. *If $|\text{Min}(R)| \geq n$, then $\alpha(\mathbb{A}\mathbb{G}_N(R)) \geq 2^{n-1}$.*

Proof. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be a subset of $\text{Min}(R)$ and $S = R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$. By Lemma 11, there exists a ring isomorphism $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$. On the other hand, if I_S, J_S are two non-adjacent vertices of $\mathbb{A}\mathbb{G}_N(R_S)$, then it is not hard to check that I, J are two non-adjacent vertices of $\mathbb{A}\mathbb{G}_N(R)$. Thus $\alpha(\mathbb{A}\mathbb{G}_N(R)) \geq \alpha(\mathbb{A}\mathbb{G}_N(R_S))$ and so by Proposition 9, we deduce that $\alpha(\mathbb{A}\mathbb{G}_N(R)) \geq 2^{n-1}$. \square

From the previous proposition, we have the following immediate corollary which shows that the finiteness of $\alpha(\mathbb{A}\mathbb{G}_N(R))$ implies the finiteness of number of the minimal prime ideals of R .

Corollary 13. *If R contains infinitely many minimal prime ideals, then the independence number of $\mathbb{A}\mathbb{G}_N(R)$ is infinite.*

Theorem 14. *For every Noetherian reduced ring R , $\alpha(\mathbb{A}\mathbb{G}_N(R)) = 2^{|\text{Min}(R)|-1}$.*

Proof. Let $\text{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$ and $S = R \setminus \bigcup_{k=1}^n \mathfrak{p}_k$. Then Lemma 11 implies that $R_S \cong R_{\mathfrak{p}_1} \times \dots \times R_{\mathfrak{p}_n}$. On the other hand, by using [9, Proposition 1.1], we deduce that every $R_{\mathfrak{p}_i}$ is a field. Thus $\alpha(\mathbb{A}\mathbb{G}_N(R)) \geq \alpha(\mathbb{A}\mathbb{G}_N(R_S)) = 2^{n-1}$, by Corollary 10. To complete the proof, it is enough to show that $\alpha(\mathbb{A}\mathbb{G}_N(R)) \leq \alpha(\mathbb{A}\mathbb{G}_N(R_S))$. To see this, let $I(x_1, x_2, \dots, x_r)$ and $J = (y_1, y_2, \dots, y_s)$ be two non-adjacent vertices of $\mathbb{A}\mathbb{G}_N(R)$. By [7, Corollary 2.4], S contains no zero-divisor and so I_S, J_S are non-trivial ideals of R_S . We show that I_S, J_S are non-adjacent vertices of $\mathbb{A}\mathbb{G}_N(R_S)$. Suppose to the contrary, $I_S J_S \subseteq \text{Nil}(R)_S = (0)$. Then for every $1 \leq i \leq r$ and $1 \leq j \leq s$, there exists $s_{ij} \in S$ such that $s_{ij} x_i y_j = 0$. Setting $t = \prod_{i,j} s_{ij}$, we have $t I J = (0)$. Since t is not a zero-divisor, we deduce that $I J = (0)$, a contradiction. Therefore, $\alpha(\mathbb{A}\mathbb{G}_N(R)) \leq \alpha(\mathbb{A}\mathbb{G}_N(R_S))$, as desired. \square

Finally as an application of the nil-graph of ideals in the ring theory we have the following corollary which shows that number of minimal prime ideals of a Noetherian reduced ring coincides number of maximal ideals of the total ring of R .

Corollary 15. *Let R be a Noetherian reduced ring. Then*

$$|\text{Min}(R)| = |\text{Max}(T(R))| = \log_2(\alpha(\mathbb{A}\mathbb{G}_N(R))).$$

Proof. Setting $\text{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$ and $S = R \setminus \bigcup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p}$, we have $T(R) \cong R_{\mathfrak{p}_1} \times \dots \times R_{\mathfrak{p}_n}$, by Lemma 11. Since every $R_{\mathfrak{p}_i}$ is a field, Corollary 10 and Theorem 14 imply that $2^{|\text{Min}(R)|-1} = 2^{|\text{Max}(T(R))|-1} = \alpha(\mathbb{A}\mathbb{G}_N(R))$. So, the assertion follows. \square

4. The Genus of Nil-Graphs of Ideals

In [3, Corollary 2.11], it is proved that for integers $q > 0$ and $g \geq 0$, there are finitely many Artinian rings R satisfying the following conditions:

- (1) $\gamma(\mathbb{A}\mathbb{G}(R)) = g$,
- (2) $|\frac{R}{\mathfrak{m}}| \leq q$ for any maximal ideal \mathfrak{m} of R .

We begin this section with a similar result for the nil-graph of ideals.

Theorem 16. *Let g and $q > 0$ be non-negative integers. Then there are finitely many Artinian rings R such that $\gamma(\mathbb{A}\mathbb{G}_N(R)) = g$ and $|\frac{R}{\mathfrak{m}}| \leq q$, for every maximal ideal \mathfrak{m} of R .*

Proof. Let R be an Artinian ring. Then [4, Theorem 8.7] implies that $R \cong R_1 \times \cdots \times R_n$, where n is a positive integer and each R_i is an Artinian local ring. We claim that for every i , $|R_i| \leq q^{\mathbb{I}(R_i)}$. Since $\gamma(\mathbb{A}\mathbb{G}_N(R)) < \infty$, we deduce that $\gamma(\mathbb{A}\mathbb{G}_N(R_i)) < \infty$, for every i . So by Remark 3 and formula for the genus of complete graphs, every R_i has finitely many ideals. Therefore, by hypothesis and [3, Lemma 2.9], we have $|R_i| \leq |\frac{R_i}{\mathfrak{m}_i}|^{\mathbb{I}(R_i)} \leq q^{\mathbb{I}(R_i)}$ and so the claim is proved. To complete the proof, it is sufficient to show that $|R|$ is bounded by a constant, depending only on g and q . With no loss of generality, suppose that $|R_1| \geq |R_i|$, for every $i \geq 2$. By the formula for the genus of complete graphs, $\frac{\mathbb{I}(R_1)-5}{12} \leq \gamma(\mathbb{A}\mathbb{G}_N(R_1)) \leq g$. Hence $|\mathbb{I}(R_1)| \leq 12g + 5$ and so

$$|R| \leq |R_1|^n \leq (q^{\mathbb{I}(R_1)})^n \leq q^{n(12g+5)}.$$

So, we are done. □

Let $\{R_i\}_{i \in \mathbb{N}}$ be an infinite family of Artinian rings such that every R_i is a direct product of 4 fields. Then it is clear that $\gamma(\mathbb{A}\mathbb{G}_N(R_i)) = 1$, for every i . So, the condition $|\frac{R}{\mathfrak{m}}| \leq q$, for every maximal ideal \mathfrak{m} of R , in the previous theorem is necessary.

Let R be a Noetherian ring. Then one may ask does $\gamma(\mathbb{A}\mathbb{G}_N(R)) < \infty$ imply that R is Artinian? The answer of this question is negative. To see this, let $R \cong S \times D$, where S is a ring with at most one non-trivial ideal and D is a Noetherian integral domain which is not a field. Then it is easy to check that $\mathbb{A}\mathbb{G}_N(R)$ is a planar graph and R is a Noetherian ring which is not Artinian.

Before proving the next lemma, we need the following notation. Let G be a graph and V' be the set of vertices of G whose degrees equal one. We use \tilde{G} for the subgraph $G \setminus V'$ and call it the *reduction* of G .

Lemma 17. $\gamma(G) = \gamma(\tilde{G})$, where \tilde{G} is the reduction of G .

Remark 18. It is well-known that if G is a connected graph of genus g , with n vertices, m edges and f faces, then $n - m + f = 2 - 2g$.

In the following, all Artinian rings, whose nil-graphs of ideals have genus at most one, are classified.

Theorem 19. *Let R be an Artinian ring. If $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$, then $|\text{Max}(R)| \leq 4$ and moreover, the following statements hold.*

- (i) *If $|\text{Max}(R)| = 4$, then $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$ if and only if R is isomorphic to a direct product of four fields.*
- (ii) *If $|\text{Max}(R)| = 3$, then $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$ if and only if $R \cong F_1 \times F_2 \times R_3$, where F_1, F_2 are fields and R_3 is an Artinian local ring with at most two non-trivial ideals.*
- (iii) *If $|\text{Max}(R)| = 2$, then $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$ if and only if either $R \cong F_1 \times R_2$, where F_1 is a field and R_2 is an Artinian local ring with at most three non-trivial ideals or $R \cong R_1 \times R_2$, where every R_i ($i = 1, 2$) is an Artinian local ring with at most one non-trivial ideal.*
- (iv) *If R is local, then $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$ if and only if R has at most 7 non-trivial ideals.*

Proof. Let $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$. First we show that $|\text{Max}(R)| \leq 4$. Suppose to the contrary, $|\text{Max}(R)| \geq 5$. By [4, Theorem 8.7], $R \cong R_1 \times \cdots \times R_5$, where every R_i is an Artinian ring. Let

$$\begin{aligned}
I_1 &= R_1 \times (0) \times (0) \times (0) \times (0); & I_2 &= (0) \times R_2 \times (0) \times (0) \times (0); \\
I_3 &= R_1 \times R_2 \times (0) \times (0) \times (0); & J_1 &= (0) \times (0) \times R_3 \times (0) \times (0); \\
J_2 &= (0) \times (0) \times (0) \times R_4 \times (0); & J_3 &= (0) \times (0) \times (0) \times (0) \times R_5; \\
J_4 &= (0) \times (0) \times R_3 \times R_4 \times (0); & J_5 &= (0) \times (0) \times (0) \times R_4 \times R_5; \\
J_6 &= (0) \times (0) \times R_3 \times (0) \times R_5; & J_7 &= (0) \times (0) \times R_3 \times R_4 \times R_5.
\end{aligned}$$

Then for every $1 \leq i \leq 3$ and every $1 \leq j \leq 7$, I_i and J_j are adjacent and so $K_{3,7}$ is a subgraph of $\mathbb{A}\mathbb{G}_N(R)$. Thus by the formula for the genus of the complete bipartite graph, we have $\gamma(\mathbb{A}\mathbb{G}_N(R)) \geq \gamma(K_{3,7}) \geq 2$, a contradiction.

(i) Let $|\text{Max}(R)| = 4$ and $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$. By [4, Theorem 8.7], $R \cong R_1 \times R_2 \times R_3 \times R_4$, where every R_i is an Artinian local ring. We show that every R_i is a field. Suppose not and with no loss of generality, R_4 contains a non-trivial ideal, say \mathfrak{a} . Set

$$\begin{aligned}
I_1 &= R_1 \times (0) \times (0) \times (0); & I_2 &= (0) \times R_2 \times (0) \times (0); & I_3 &= R_1 \times R_2 \times (0) \times (0); \\
I_4 &= R_1 \times R_2 \times (0) \times \mathfrak{a}; & J_1 &= (0) \times (0) \times R_3 \times (0); & J_2 &= (0) \times (0) \times (0) \times R_4;
\end{aligned}$$

$$J_3 = (0) \times (0) \times (0) \times \mathfrak{a}; \quad J_4 = (0) \times (0) \times R_3 \times R_4; \quad J_5 = (0) \times (0) \times R_3 \times \mathfrak{a}.$$

It is clear that every I_i , $1 \leq i \leq 4$, is adjacent to J_j , $1 \leq j \leq 5$, and so $K_{4,5}$ is a subgraph of $\mathbb{A}\mathbb{G}_N(R)$. Thus by the formula for the genus of the complete bipartite graph, we have $\gamma(\mathbb{A}\mathbb{G}_N(R)) \geq \gamma(K_{4,5}) \geq 2$, a contradiction. Conversely, assume that $R \cong F_1 \times F_2 \times F_3 \times F_4$, where every F_i is a field. We show that $\gamma(\mathbb{A}\mathbb{G}_N(R)) = 1$. By Lemma 17, it is enough to prove that $\gamma(\widetilde{\mathbb{A}\mathbb{G}_N(R)}) = 1$. We know that $\mathbb{A}\mathbb{G}_N(R)$ has 4 vertices of degree 6 and 6 vertices of degree 3. So, $\widetilde{\mathbb{A}\mathbb{G}_N(R)}$ has $n = 10$ vertices and $m = 21$ edges. Also, it is not hard to check that $\mathbb{A}\mathbb{G}_N(R)$ has $f = 11$ faces. Now, Remark 18 implies that $\gamma(\widetilde{\mathbb{A}\mathbb{G}_N(R)}) = 1$.

(ii) Let $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$ and $R \cong R_1 \times R_2 \times R_3$, where every R_i is an Artinian local ring. We show that at least two of the three rings R_1 , R_2 and R_3 are fields. Suppose not and with no loss of generality, \mathfrak{b} and \mathfrak{c} are non-trivial ideals of R_2 and R_3 , respectively. Set

$$I_1 = R_1 \times (0) \times (0); \quad I_2 = (0) \times R_2 \times (0); \quad I_3 = R_1 \times R_2 \times (0); \quad I_4 = R_1 \times \mathfrak{b} \times (0);$$

$$J_1 = (0) \times \mathfrak{b} \times (0); \quad J_2 = (0) \times (0) \times \mathfrak{c}; \quad J_3 = (0) \times \mathfrak{b} \times \mathfrak{c};$$

$$J_4 = (0) \times (0) \times R_3; \quad J_5 = (0) \times \mathfrak{b} \times R_3.$$

It is clear that every I_i , $1 \leq i \leq 4$, is adjacent to J_j , $1 \leq j \leq 5$, and so $K_{4,5}$ is a subgraph of $\mathbb{A}\mathbb{G}_N(R)$. Thus by the formula for the genus of the complete bipartite graph, we have $\gamma(\mathbb{A}\mathbb{G}_N(R)) \geq \gamma(K_{4,5}) \geq 2$, a contradiction. Thus with no loss of generality, we can suppose that $R \cong F_1 \times F_2 \times R_3$, where F_1 and F_2 are fields and R_3 is an Artinian local ring. Now, we prove that R_3 has at most two non-trivial ideals. Suppose to the contrary, \mathfrak{a} , \mathfrak{b} and \mathfrak{c} are three distinct non-trivial ideals of R_3 . Let

$$I_1 = (0) \times (0) \times R_3; \quad I_2 = (0) \times (0) \times \mathfrak{a}; \quad I_3 = (0) \times (0) \times \mathfrak{b}; \quad I_4 = (0) \times (0) \times \mathfrak{c};$$

$$J_1 = F_1 \times (0) \times (0); \quad J_2 = (0) \times F_2 \times (0); \quad J_3 = F_1 \times F_2 \times (0);$$

$$J_4 = F_1 \times F_2 \times \mathfrak{a}; \quad J_5 = F_1 \times F_2 \times \mathfrak{b}; \quad J_6 = F_1 \times F_2 \times \mathfrak{c}.$$

Clearly, every I_i , $1 \leq i \leq 4$, is adjacent to J_j , $1 \leq j \leq 6$, and so $K_{4,6}$ is a subgraph of $\mathbb{A}\mathbb{G}_N(R)$. Thus by the formula for the genus of the complete bipartite graph, we have $\gamma(\mathbb{A}\mathbb{G}_N(R)) \geq \gamma(K_{4,6}) \geq 2$, a contradiction. Conversely, let $R \cong F_1 \times F_2 \times R_3$, where F_1 and F_2 are fields and R_3 be a ring with two non-trivial ideals \mathfrak{c} and \mathfrak{c}' . Set

$$I_1 = (0) \times (0) \times R_3; \quad I_2 = (0) \times (0) \times \mathfrak{c}; \quad I_3 = (0) \times (0) \times \mathfrak{c}';$$

$$J_1 = F_1 \times (0) \times (0); \quad J_2 = (0) \times F_2 \times (0); \quad J_3 = F_1 \times F_2 \times (0).$$

Then for every $1 \leq i, j \leq 3$, we have $I_i J_j = (0)$. Hence $\gamma(\mathbb{A}\mathbb{G}_N(R)) \geq \gamma(K_{3,3}) \geq 1$. However, in this case, $\mathbb{A}\mathbb{G}_N(R)$ is a subgraph of $\mathbb{A}\mathbb{G}_N(F_1 \times F_2 \times F_3 \times F_4)$ (in which every F_i is a field). Therefore, by (i), $\gamma(\mathbb{A}\mathbb{G}_N(R)) = 1$. If R_3 contains at most one non-trivial ideal, then it is not hard to check that $\mathbb{A}\mathbb{G}_N(R)$ is a planar graph. This completes the proof of (ii).

(iii) Assume that $\gamma(\mathbb{A}\mathbb{G}_N(R)) < 2$ and $R \cong R_1 \times R_2$, where R_1 and R_2 are Artinian local rings. We prove the assertion in the following two cases:

Case 1. $R \cong F_1 \times R_2$, where F_1 is a field and R_2 is an Artinian local ring. In this case, we show that R_2 has at most three non-trivial ideals. Suppose to the contrary, R_2 has at least four non-trivial ideals. Then for every two non-zero ideals $I_2 \neq R_2$ and J_2 of R_2 , the vertices $F_1 \times I_2$ and $(0) \times J_2$ are adjacent and so $K_{4,5}$ is a subgraph of $\mathbb{A}\mathbb{G}_N(R)$. Thus by the formula for the genus of the complete bipartite graph, we have $\gamma(\mathbb{A}\mathbb{G}_N(R)) \geq \gamma(K_{4,5}) \geq 2$, a contradiction.

Case 2. Neither R_1 nor R_2 is a field. We prove that every R_i has at most one non-trivial ideal. Suppose not and with no loss of generality, R_2 has two distinct non-trivial ideals. Then every ideal of the form $R_1 \times J$ is adjacent to every ideal of the form $I \times K$, where I and J are proper ideals of R_1 and R_2 , respectively, and K is an arbitrary ideal of R_2 . So $\gamma(\mathbb{A}\mathbb{G}_N(R)) \geq \gamma(K_{3,7}) \geq 2$, a contradiction.

Conversely, if $R \cong F_1 \times R_2$, where F_1 is a field and R_2 is an Artinian local ring with $n \leq 3$ non-trivial ideals, then one can easily show that

$$\gamma(\mathbb{A}\mathbb{G}_N(R)) = \begin{cases} 1; & n = 2, 3 \\ 0; & n = 1. \end{cases}$$

Now, suppose that $R \cong R_1 \times R_2$, where R_1 and R_2 are Artinian local rings with one non-trivial ideal. Then it is not hard to show that $\gamma(\mathbb{A}\mathbb{G}_N(R)) = 1$. This completes the proof of (iii).

(iv) This follows from the formula of genus for the complete graphs and Remark 3.

□

From the proof of the previous theorem, we have the following immediate corollary.

Corollary 20. *Let R be an Artinian ring. Then $\mathbb{A}\mathbb{G}_N(R)$ is a planar graph if and only if $|\text{Max}(R)| \leq 3$ and R satisfies one of the following conditions:*

- (i) R is isomorphic to the direct product of three fields.

- (ii) $R \cong F_1 \times R_2$, where F_1 is a field and R_2 is an Artinian local ring with at most one non-trivial ideal.
- (iii) R is a local ring with at most four non-trivial ideals.

We close this paper with the following example.

Example 21. (i) Suppose that $R \cong \frac{\mathbb{Z}_6[x]}{(x^m)}$, where $m \geq 2$. Let $I_1 = (3)$, $I_2 = (3x)$, $I_3 = (3x + 3)$, $J_1 = (2)$, $J_2 = (4)$, $J_3 = (2x)$, $J_4 = (4x)$, $J_5 = (2x + 2)$, $J_6 = (4x + 2)$ and $J_7 = (2x + 4)$. Then one can check that these ideals are distinct vertices of $\mathbb{AG}_N(R)$. Also, every I_i ($1 \leq i \leq 3$) is adjacent to every J_k ($1 \leq k \leq 7$). Thus $K_{3,7}$ is a subgraph of $\mathbb{AG}_N(R)$ and so the formula of genus for the complete bipartite graphs implies that $\gamma(\mathbb{AG}_N(R)) \geq 2$.

(ii) Let $R \cong \frac{\mathbb{Z}_4[x]}{(x^3)}$. Set $I_1 = (2x)$, $I_2 = (2x^2)$, $I_3 = (2x + 2x^2)$, $J_1 = (2)$, $J_2 = (2 + x^2)$, $J_3 = (2 + 2x^2)$, $J_4 = (2 - x^2)$, $J_5 = (2 + 2x)$, $J_6 = (2 + 2x + x^2)$ and $J_7 = (2 + 2x + 2x^2)$. Similar to (i), one can show that every I_i is adjacent to every J_k and so $\gamma(\mathbb{AG}_N(R)) \geq 2$.

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